Differential Equations (Formation & Solution)

Introduction:

Any relation between an independent variable x, a dependent variable y, and one or more of the derived functions

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}, \ldots$$

is called an ordinary differential equation.

Examples: (i) $\frac{dy}{dx} = m$

(ii) $\frac{d^2y}{dx^2} = -a^2y$

(iii) $x\,dy - y\,dx + 2xy\,dy = 0$

The order of a differential equation is the order of the highest ordered derivative involved in the equation.

The degree of a differential equation is the power of the highest ordered derivative involved in the equation after it has been made rational and integral as far as derivatives are concerned. That means if an ordinary differential equation can be written as a polynomial in the unknown function and its derivatives, then its degree is the power to which the highest ordered derivative is raised.

Example ① $y = x\dfrac{dy}{dx} + c\dfrac{dx}{dy}$

$$\Rightarrow y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + c$$

This equation is of first order and second degree.

Example ②

$$\left\{1+\frac{d^2y}{dx^2}\right\}^{3/2} = a\frac{d^2y}{dx^2}$$

$$\Rightarrow \left(1+\frac{d^2y}{dx^2}\right)^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$$

This is of second order and third degree differential equation.

Remark: → An ordinary differential equation of order n can be exhibited

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots, \frac{d^ny}{dx^n}\right) = 0 \quad — (*)$$

Definition: → If a differential equation is of first degree in the dependent variable y and its derivative (consequently, there cannot be any term involving y and its derivatives), then it is called a linear differential equation; otherwise, it is non-linear.

A general form of a linear ordinary differential equation of order n in the unknown function y (or the dependent variable y) and the independent variable x is

$$f_0(x)\frac{d^ny}{dx^n}+f_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots\cdots+f_{n-1}(x)\frac{dy}{dx}+f_n(x)\cdot y = f(x) \qquad (1)$$

Where the functions $f_i(x)$, $i=0,1,2,\cdots,n$ & $f(x)$ are presumed known and depend only on the variable x.

Differential equation that cannot be put into this form are said to be <u>non-linear</u>.

<u>Example (1)</u> $\frac{dy}{dx}=1+x^2y^2$

is of first degree but non-linear ($\because y^2$ is present)

<u>Formation of ordinary differential equation :→</u>

If we have an equation containing n <u>arbitrary</u> constants, then by differentiating it n times we shall get (n+1) equations ~~along~~ altogether. If we eliminate the n constants from the (n+1) equations, we shall get a differential equation of the <u>nth</u> order i.e. an equation of the form

$$f_1\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots\cdots, \frac{d^ny}{dx^n}\right)=0$$

Example ①: → Let $y^2 = 4ax$ ——— ①

Differentiating ①, w.r.to x, we have

$$2y\frac{dy}{dx} = 4a \quad \text{———} \quad ②$$

eliminating constant a from equations ① & ②

$$\therefore 2y\frac{dy}{dx} = \frac{y^2}{x}$$

$$\Rightarrow 2x\frac{dy}{dx} = y$$

Which is a differential equation of first order

Example ②

Let $y = mx + c$ ——— ①

Differentiating ① w.r.to x, we have

$$\frac{dy}{dx} = m \quad \text{———} \quad ②$$

Again differentiating ②, we have

$$\frac{d^2y}{dx^2} = 0 \quad \text{———} \quad ③$$

Which is a differential equation of second degree.

Solution of a differential Equation :→

The general ordinary differential equation of the nth order as given by

$$g\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots, \frac{d^ny}{dx^n}\right) = 0 \quad \text{——} (1)$$

i.e; $g(x, y, y', y'', \ldots, y^{(n)}) = 0$ —— (2)

Let us assume that we can solve equation (2) for $y^{(n)}$, that is equation (2) can be written in the form.

$$y^{(n)} = f(x, y, y', y'', \ldots, y^{(n-1)}) \quad \text{——} (3)$$

If a given function $y = \phi(x)$ satisfies an equation like (2) & (3). All that is necessary is to compute the derivatives of y and to show that $y = \phi(x)$ and its derivatives, when substituted in the equation, reduce it to an identity in x. If such function y exists, we call it a solution of equation (2) or (3).

Definition :→ A real or complex valued function $y = \phi(x)$ defined on an interval I is called a <u>solution</u> or an <u>integral</u> of the differential equation $g(x, y, y', \ldots\ldots, y^{(n)}) = 0$ if $\phi(x)$ is n times differentiable & if $x, \phi(x), \phi'(x), \ldots, \phi^{n}(x)$ satisfy this equation for all x in I.

Example ① Show that for any constant c, the function $y(x) = ce^{x}$, $x \in \mathbb{R}$ is a solution of

$$\frac{dy}{dx} = y, \; x \in \mathbb{R}. \qquad \text{—— ①}$$

Solution:→ Here I is $\mathbb{R}$ itself.

For any $x \in \mathbb{R}$, we know that

$$\frac{dy}{dx} = \frac{d}{dx}(ce^{x}) = c\frac{d}{dx}(e^{x}) = ce^{x} = y$$

Which shows that y satisfies ①

Example ② Show that $y(x) = e^{ix}$, $x \in \mathbb{R}$ is a solution of

$$\frac{d^2y}{dx^2} + y = 0, \; x \in \mathbb{R}.$$

Solution:→ We have,

$$\frac{dy}{dx} = \frac{d}{dx}(e^{ix}) = ie^{ix}$$

$$\& \; \frac{d^2y}{dx^2} = \frac{d}{dx}(ie^{ix}) = i^2e^{ix} = -e^{ix} = -y(x)$$

Thus $\frac{d^2y}{dx^2} + y = 0$

Definition:→ The solution of the nth order differential equation which contains n arbitrary constants is called its <u>general Solution</u>.

Definition:→ Any solution which is obtained from the general solution by giving particular values to the arbitrary constants is called a <u>particular solution</u>.

Example ③ $\because y = A\cos 2x + B\sin 2x$, involving two arbitrary constants a and b, is the general solution of the second order equation $\frac{d^2y}{dx^2} + 4y = 0$

Whereas, $y = 2\sin 2x + \cos 2x$ is its particular solution taking $a=1$ & $b=2$

In some cases there may be further solutions of ~~the~~ a given equation which cannot ~~by~~ be obtained by assigning a definite value to the arbitrary constant in the general solution. Such a solution is called a singular solution of the equation.

Example ④ The equation

$$y'^2 - xy' + y = 0 \quad -①$$

has the general solution $y = cx - c^2$.

A further solution of equation ① is $y = \frac{x^2}{4}$.

Since this solution cannot be obtained by assigning a definite value c in the general solution, it is a singular solution of ①